## GROUP THEORY 2024 - 25, SOLUTION SHEET 7

**Exercise 1.** Review the lecture and understand/fill in the gaps in the proofs.

**Exercise 2.** By the correspondence theorem, normal subgroups of $G$ that contain $H$ are in bijection with normal subgroups of $G/H$. This proves both implications of the claim.

**Exercise 3.** A composition series is given by

$$0 = 12\mathbb{Z}/12\mathbb{Z} \trianglelefteq 6\mathbb{Z}/12\mathbb{Z} \trianglelefteq 3\mathbb{Z}/12\mathbb{Z} \trianglelefteq \mathbb{Z}/12\mathbb{Z}$$

with composition factors

$$\{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}\}.$$

The composition series is not unique, for example here is another one

$$0 = 12\mathbb{Z}/12\mathbb{Z} \trianglelefteq 6\mathbb{Z}/12\mathbb{Z} \trianglelefteq 2\mathbb{Z}/12\mathbb{Z} \trianglelefteq \mathbb{Z}/12\mathbb{Z}$$

They have the same composition factors by a theorem of the lectures.

**Exercise 4.** Let $V_4 = \{1, (12)(34), (13)(24), (14)(23)\}$ be the Klein four-group. Notice that it is precisely the subgroup of $A_4$ elements of order 2. Since for all $\sigma \in A_4$ and $x \in V_4$ we have $(\sigma x \sigma^{-1})^2 = 1$, this prove that $\sigma x \sigma^{-1}$ has order 2 and hence belong to $V_4$. This shows that $V_4$ is normal in $A_4$. It follows that

$$0 \trianglelefteq \mathbb{Z}/2\mathbb{Z} = \langle (12)(34) \rangle \trianglelefteq V_4 \trianglelefteq A_4$$

is a composition series. Its composition factors are

$$\{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}\}$$

since those are the only groups with the required cardinalities. Since $A_4 \trianglelefteq S_4$ is normal, we can extend it to a composition series

$$0 \trianglelefteq \mathbb{Z}/2\mathbb{Z} = \langle (12)(34) \rangle \trianglelefteq V_4 \trianglelefteq A_4 \trianglelefteq S_4$$

with composition factors

$$\{\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\}.$$

**Exercise 5.** By the properties of semi-direct products, we have an exact sequence:

$$1 \to G \to G \rtimes_\varphi H \to H \to 1.$$

Then it follows from Proposition 22 of the notes that the composition factors of $G \rtimes_\varphi H$ are just the compositions factors of $G$ and the composition factors of $H$.

**Exercise 6.** (1) By exercise 5 of last week, we know that we can write
$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \mathbb{Z}/p_2^{a_2}\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k^{a_k}\mathbb{Z}.$$
Hence a composition series is given by the following:
$$0 \trianglelefteq \mathbb{Z}/p_1\mathbb{Z} \trianglelefteq \mathbb{Z}/p_1^2\mathbb{Z} \trianglelefteq \mathbb{Z}/p_1^3\mathbb{Z} \trianglelefteq \ldots \trianglelefteq \mathbb{Z}/p_1^{a_1}\mathbb{Z} \trianglelefteq \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \mathbb{Z}/p_2\mathbb{Z} \trianglelefteq$$
$$\mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \mathbb{Z}/p_2^2\mathbb{Z} \trianglelefteq \ldots \trianglelefteq \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \mathbb{Z}/p_2^{a_2}\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k^{a_k}\mathbb{Z}$$
which has length $a_1 + a_2 + \ldots + a_k$. The composition factors consist of $a_i$-times $\mathbb{Z}/p_i\mathbb{Z}$ for all $1 \leq i \leq k$.

(2) Let $n \in \mathbb{N}$. Using proposition 19 of the lectures, we know that $G = \mathbb{Z}/n\mathbb{Z}$ has a composition series
$$0 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_k = G.$$
Since $G$ is abelian, so are his subgroups. Hence the composition factors $G_{i+1}/G_i$ are finite simple abelian groups, i.e. they are cyclic of prime order (as explained in the lectures). It follows that
$$n = |G| \cong |G/G_{k-1}| \times |G_{k-1}| \cong |G/G_{k-1}| \times |G_{k-1}/G_{k-2}| \times |G_{k-2}|$$
$$\cong \prod_{i=0}^{k-1} |G_{i+1}/G_i|$$
which is a product of primes. By the Jordan Hölder theorem, the composition factors $G_{i+1}/G_i$ are unique (up to permuting the factors), which shows that such a decomposition of $n$ as a product of primes is unique.

**Exercise 7.** By exercise 7 of sheet 4 we have an isomorphism $D_{2n} \cong \mathbb{Z}/n\mathbb{Z} \times_\varphi \mathbb{Z}/2\mathbb{Z}$. Hence we have a short exact sequence
$$0 \to \mathbb{Z}/n\mathbb{Z} \to D_{2n} \to \mathbb{Z}/2\mathbb{Z} \to 0$$
and so by the lectures we know that $D_{2n}$ has a composition series given by attaching a composition series of $\mathbb{Z}/n\mathbb{Z}$ with one of $\mathbb{Z}/2\mathbb{Z}$. The previous exercise gives us such composition series. Moreover, exercise 5 tells us that the composition factors is the union of the factors of those two groups.

**Exercise 8.** Suppose by contradiction that we have the existence of a proper normal subgroup $H \trianglelefteq G$. Then if we let $G_0 := 1$, there exists $n \in \mathbb{N}$ such that $G_n \subseteq H$ and $G_{n+1} \nsubseteq H$. However, we then have that $G_{n+1} \bigcap H$ is a proper normal subgroup of $G_{n+1}$, which contradicts the assumption. Consider the inclusions $A_5 \subset A_6 \subset \ldots$ of the alternating groups, all of which are simple. Then
$$\bar{A} = \bigcup_{i=5}^{\infty} A_i$$
is infinite and simple.

**Exercise 9.** (1) See the proof of propositions $20, 21, 22$ and the Jordan-Holder theorem.

(2) We have a short exact sequence

$$1 \to K \to G \to G/K \to 1$$

By the previous point, $\text{length}(G) = \text{length}(K) + \text{length}(G/K)$, but as $K$ is a proper subgroup of $G$, we have that $G/K$ is not trivial and thus of length strictly greater than 0. This implies that $\text{length}(G) > \text{length}(K)$.

(3) If we have a strict chain

$$1 \trianglelefteq G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \dots$$

made up of normal subgroups of $G$, we can apply (2) to get

$$0 < \text{length}(G_0) < \text{length}(G_1) < \dots$$

Thus, any such chain must be finite and have length at most equal to $\text{length}(G) + 1$.

(4) Let us prove each of the two implications:

" $\implies$ " Observe that the reasoning of (3) still holds, even if the chains described in $(a)$ and $(b)$ are not made up of normal subgroups of $G$.

" $\impliedby$ " Clearly, $G$ is normal in $G$. If $G$ is simple, we are already done. If not, pick a normal subgroup $H_0 \trianglelefteq G$. If $H_0$ is maximal in $G$, we stop. If not, we continue iterating this process by choosing at each time a normal subgroup $H_i$ of $G$ such that $H_{i-1} \trianglelefteq H_i$ with strict inclusion. By assumption $(b)$, this process must terminate at some $H_n$ for $n$ a positive integer. Set $G_1 = H_n \trianglelefteq G$. One can check that $G_1$ is maximal in $G$, if not the above process would have not terminated. Observe that $G_1$ is also normal in $G$, by construction, Thus, we can inductively apply the same reasoning as above to obtain a descending normal chain in which each inclusion is maximal:

$$G \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \dots$$

By assumption $(a)$ such a chain must stabilize at some $G_n$ and by construction of the $G_i$ we must have that $G_n = 1$. We have thus obtained a composition series for $G$, so $G$ has finite length.